2022 John O'Bryan Mathematics Competition
5-Person Team Test

Abbreviated Instructions: Answer each of the following questions using separate sheet(s) of paper. Questions will not be scored without the following two items:

- Place your team code in the upper right corner of each page that will be turned in.
- Place question numbers in the upper left corner of each page that will be turned in.

Questions are equally weighted. Teams must show complete solutions (not just answers) to receive credit. More specific instructions are read verbally at the beginning of the test.

1. For **all natural numbers n**, let $S(n)$ be the sum of the digits of n plus the number of digits of n. For instance, $S(125) = 1 + 2 + 5 + 3 = 11$. Note that the first digit of n, when reading from left to right, cannot be zero.

   a. Determine S(12408)

   b. Determine all numbers n such that $S(n) = 4$

   c. Determine whether or not there exists a natural number m such that $S(m) - S(m + 1) > 50$. Provide a clear justification for your answer.

2. Suppose line J in the xy-plane is given by the equation $5y + (2c - 4)x - 10c = 0$, where c is some real number. Furthermore, suppose line J intersects the rectangle with vertices O(0,0), A(0,6), B(10,6), and C(10,0) at point M on line segment OA and point N on line segment BC.

   a. Show that $1 \leq c \leq 3$

   b. Show that the area of quadrilateral AMNB is one-third the area of rectangle OABC

   c. Find the equation, in terms of c of the line parallel to J that has the following characteristics: (1) the line intersects segment OA at point P, (2) the line intersects segment BC at point Q, and (3) quadrilaterals AMNB, MNQR, and RQCO all have the same area.

3. Suppose $g(x)$ is the quadratic function $g(x) = x^2 - ax + b$, where a and b are **natural numbers.**

   a. If $a = b = 2$, find the set of real roots of the expression $g(x) - x$.

   b. If $a = b = 2$, find the set of real roots of the expression $g(g(x)) - x$.

   c. Find the number of pairs of **natural numbers** (a, b) where $1 \leq a \leq 2022$, $1 \leq b \leq 2022$, and every root of the expression $g(g(x)) - x$ is an integer.

4. Jaden takes a mathematics test consisting of 100 questions, where the answer to each question is either TRUE or FALSE. For every five consecutive questions on the test, the answers to exactly three of the questions are TRUE. If the answers to Question 1 and Question 100 are both FALSE:

   a. Find the number of questions on the test for which the correct answer is TRUE.

   b. Find the correct answer to the sixth question on the test.

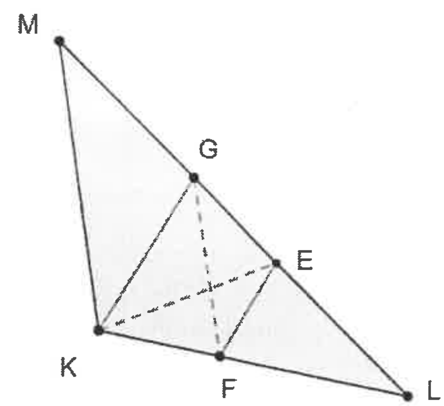
   c. Explain how Jaden can correctly answer ALL 100 questions on the test.

5. Suppose quadrilateral STRV is an isosceles trapezoid, with ST = 5 cm, RV = 5 cm, TR = 2 cm, and SV = 8 cm.

   a. Find the lengths of the altitude of STRV to the parallel bases and the diagonal SR.

   b. For the isosceles trapezoid in part (a), what is the exact value of the **cosine** of ∠RTS?

   c. In triangle KLM shown, points G and E are points on segment LM so that ∠MKG ≅ ∠GKE ≅ ∠EKL. Also, point F is located on segment KL so that segment GF is parallel to segment KM. If quadrilateral KFEG is an isosceles trapezoid and the measure of ∠KLM is 84°, find the measure of ∠MKG.



6. If M is a **natural number**, then a "nice division" of M is a partition of the set {1, 2, … , M} into two disjoint, non-empty subsets A_1 and A_2 such that the **sum** of the numbers in A_1 is equal to the **product** of the numbers in A_2. If M = 8, for instance, then A_1 = {2, 4, 5, 6, 7} and A_2 = {1, 3, 8} is a "nice division" of M.

   a. Find a "nice division" of M = 7.

   b. Find a value of M such that there are two distinct "nice divisions" of M.

   c. A curious student claims that for every natural number M ≥ 5, there is a "nice division" of M. Show or explain why the student is correct.

2022 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. Determine which of the following five statement(s) is sufficient to deduct that $x > y$.

   A) $x + 1 = y$ B) $x + 2.2 = y$ C) $x - 1.3 = y$ D) $xy > 0$ E) $xy < 0$

2. Jadyn just completed a 4-day trip. On the first day she drove 531 miles; on the second day, 615 miles; on the third day, 704 miles; on the fourth day, k miles. If Jadyn drove an average of 622.5 miles per day for the 4-day trip, determine the value of k.

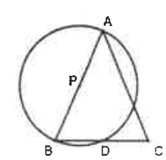
3. A rectangle is half as wide as it is long and has a perimeter of x units. The numeric area of this rectangle can be written as the expression kx^2. Determine the value of k. Express your answer as an integer or as a common or improper fraction reduced to lowest terms.

4. A circle is inscribed within the region that is the solution of $\begin{cases} x \geq 2 \\ x \leq 6 \\ y \geq 1 \\ y \leq 5 \end{cases}$. This circle can be represented algebraically by $(x - h)^2 + (y - k)^2 = r^2$. Determine the ordered triple (h, k, r^2).

5. Let $x^2 - 4y^2 = 30$ and $x - 2y = 5$. Determine the value of $(x + 2y)$.

6. The sum of twice a number and three times a second number is 16. The difference between the two numbers is 3. If the first number is greater than the second number, determine the sum of the two numbers.

7. Circle P has diameter $\overline{AB}$. ΔABC is isosceles with base $\overline{BC}$ intersecting the circle at point D. $AC = 4$ and $DC = 1$. Determine the numeric area of ΔABC. Give your answer as a radical expression (in the form $a\sqrt{b}$), where b is a whole number as small as possible.



8. In a circle with center C, minor arc $\frown AB$ has length $\frac{8\pi}{9}$. $\angle ACB = 40°$. Determine the radius of the circle C.

9. Let $k = \sqrt{110 + \sqrt{110 + \sqrt{110 + \sqrt{110 \ldots}}}}$. Determine the exact value of k.

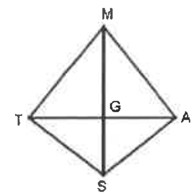
10. Let b and c be integers with $g(x) = x^2 + bx + c$ and $f(x) = x^2 + cx + b$. Determine the sum $(b + c)$ when $g(c) = f(b)$ and $c \neq b$.

11. A square with numeric area k is inscribed in a semicircle (two vertices of the square lie on the semicircle and the side opposite the two vertices lies on the diameter.) A second square with numeric area w is inscribed in a full congruent circle (same radius as the semicircle). Determine the ratio $k : w$. Give your answer as a ratio where both k and w are whole numbers that are as small as possible.

12. Determine the number of diagonals that can be drawn in a convex polygon that has 100 sides.

13. Let a and b be such that $a^2b - 3a - 2ab^2 + 6b = 32$. If a is 4 more than twice b, determine the product (ab).

14. For all x in the expression's natural domain, the expression $\left[\frac{2}{x+3} + \frac{x-1}{x+2} - \frac{4x+5}{x^2+5x+6}\right]^{(-1)}$ can be written as a single simplified rational expression. Determine the numerator of this simplified rational expression. Express you answer as an algebraic expression in the form of an irreducible polynomial.

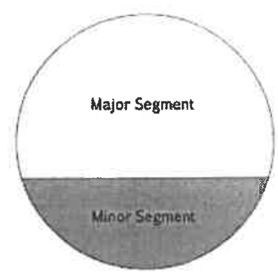


15. MAST is a kite with $TM = TS$ (note: drawing not to scale). $TG = 2y - 2$, $GA = y + 3$, $MG = 2y$, and $GS = y + 6$. Determine the length of $\overline{TS}$. Give your answer in the form where $a\sqrt{b}$, where a and b are integers and b is as small as possible.

16. Line l, with equation $3x - 4y - 4 = 0$, intersects $\odot O$, with equation $(x-1)^2 + (y-k)^2 = 25$, in exactly one point P. Determine *the sum* of all possible value(s) of k. Express your answer as an integer or as a common or improper fraction reduced to lowest terms.

17. Determine the numerical area of the triangular region enclosed by the system $\begin{cases} y = 4x - 2 \\ y = -4x + 14 \\ y = \frac{4}{3}x - 2 \end{cases}$.

18. A square with area S and an isosceles right triangle with area T have the same perimeter. Determine the larger of the two ratios $\frac{S}{T}$ and $\frac{T}{S}$. Express your exact answer as a decimal rounded to the nearest thousandth.



19. A minor segment (see figure) of a circle with radius 12 is determined by a 120° arc. Determine the area of this segment of the circle. Express your answer as a decimal rounded to the nearest thousandth.

20. Rectangle $RECT$ has diagonals that meet at point Z. $RZ = 650$, and the numeric area of rectangle $RECT$ is 739200. Determine the perimeter of rectangle $RECT$.

Name: ______________________ Team Code: _______

2022 John O'Bryan Mathematical Competition
Freshman/Sophomore Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. ______________________

2. ______________________

3. ______________________

4. ______________________

5. ______________________

6. ______________________

7. ______________________

8. ______________________

9. ______________________

10. ______________________

11. ______________________

12. ______________________

13. ______________________

14. ______________________

15. ______________________

16. ______________________

17. ______________________

18. ______________________

19. ______________________

20. ______________________

Name: ______ANSWERS______ Team Code: _______

2022 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

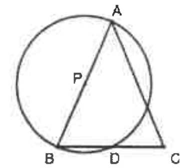
1. C
2. 640
3. $\frac{1}{18}$ Must be this reduced common fraction.
4. (4, 3, 4) Must be this ordered triple.
5. 6
6. 7
7. $\sqrt{15}$ Must be this exact answer.
8. 4
9. 11
10. -1
11. 2:5 Must be this exact ratio.
12. 4850
13. 11
14. $x+3$ or $3+x$ Must be this algebraic expression.
15. $2\sqrt{61}$ Must be this exact answer.
16. $\frac{1}{2}$ Must be this reduced common fraction.
17. 8
18. 1.457 Must be this decimal.
19. 88.443 Must be this decimal.
20. 3560

Awards Lists and Solutions to the Team Competition may be found at https://inside.nku.edu/artsci/departments/math/competitions/obryan.html

2022 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. Circle P has diameter $\overline{AB}$. ΔABC is isosceles with base $\overline{BC}$ intersecting the circle at point D. $AC = 4$ and $DC = 1$. Determine the numeric area of ΔABC. Give your answer as a radical expression (in the form $a\sqrt{b}$), where b is a whole number as small as possible.

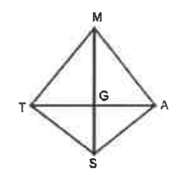


2. Determine the sum of all distinct values(s) for x such that $\sqrt{2x-12} = x - 10$.

3. Determine the constant term when the expansion of $\left(x - \frac{2}{x}\right)^4$ is expanded and completely simplified.

4. Determine the area enclosed by the graph of the inequality $|3x-12| + |2y+4| \leq 3$. Express your answer as an integer or as a common or improper fraction reduced to lowest terms.

5. MAST is a kite with $TM = TS$ (note: drawing not to scale). $TG = 2y - 2$, $GA = y + 3$, $MG = 2y$, and $GS = y + 6$. Determine the length of $\overline{TS}$. Give your answer in the form where $a\sqrt{b}$, where a and b are integers and b is as small as possible.



6. Let n be a positive integer such that $\frac{n!}{(n-2)!} + (2n-4)^2 - \frac{n!}{(n-1)!} = 0$. Determine the value(s) of n.

7. Determine the numerical area of the triangular region enclosed by the system $\begin{cases} y = 4x - 2 \\ y = -4x + 14 \\ y = \frac{4}{3}x - 2 \end{cases}$.

8. Determine the product of all possible value(s) for x such that $|x+2| + |x| = 8$.

9. $i = \sqrt{-1}$ and $A = i + i^2 + i^3 + i^4 + i^5 + \cdots + i^{2022}$. Determine the exact value of A.

10. The graph of an ellipse with center at (7, 2) is tangent to the x-axis at point (7, 0) and the y-axis at point (0, 2). The exact area of this ellipse may be expressed as $k\pi$. Find the value of k.

11. Determine the geometric mean of 3, 27, 72.

12. The bullet train traveled at r miles per hour for t hours and was 143 miles short of the train station at the estimate time of arrival. The next day, on the same run starting at the same time and same place, the engineer increased the rate of travel by 13 miles per hour and reached the station at the same estimated time of arrival. Determine the number of hours it took the train to reach the station on the second trip.

13. Determine the value of k such that $\frac{16^{(k)} \times 8^{(k+3)}}{32^{(k-1)} \times 4^{(k)} \times 2^{(4k)}} = 32^{(4)}$. Express your answer as an integer or as a common or improper fraction reduced to the lowest terms.

14. In a geometric sequence, $a_1 = \frac{5}{4}$ and $a_4 = -10$. Determine the value of a_3. Express your answer as an integer or as a common or improper fraction reduced to lowest terms.

15. Rectangle $RECT$ has diagonals that meet at point Z. $RZ = 650$, and the numeric area of rectangle $RECT$ is 739200. Determine the perimeter of rectangle $RECT$.

16. Let $\lim_{n \to \infty} \left(\ln \left(\frac{n^2 + \cos n}{n^2 + 167} \right) \right) = k$. Determine the exact value of k if it exists. Express your answer as an integer or as a common or improper fraction reduced to lowest terms of "DNE" if the limit does not exist.

17. Let $A = \log_2 3$, $B = \log_2 5$, $C = \log_2 7$, $D = \log_2 11$. Then $\log_4 \left(\frac{31500}{1331} \right) = aA + bB + cC + dD + f$ for real numbers $a, b, c, d,$ and f. Determine the sum $(a + b + c + d + f)$. Express your answer as an integer of as a common or improper fraction reduced to lowest terms.

18. Jack eats lunch at McDonalds or Burger King every week day according to the following pattern. When Jack eats at McDonalds, he eats at McDonalds the next day 70% of the time. When Jack eats at Burger King, he eats at Burger King the next day 80% of the time. Jack ate lunch at Burger King Monday. According to his pattern, determine the probability Jack will eat lunch at McDonalds on Friday, four days later. Express your answer as a decimal.

19. A square with numeric area k is inscribed in a semicircle (two vertices of the square lie on the semicircle and the side opposite the two vertices lies on the diameter.) A second square with numeric area w is inscribed in a full congruent circle (same radius as the semicircle). Determine the ratio $k\!:\!w$. Give your answer as a ratio where both k and w are whole numbers that are as small as possible.

20. $i = \sqrt{-1}$. $f(x) = x^2 + kx + w$ has real coefficients k and w. The zeros of $f(x)$ are r_1 and r_2 such that $r_1 - r_2 = 2i$ and $\frac{1}{r_1} + \frac{1}{r_2} = \frac{3}{5}$. Determine the sum of all possible value(s) for k and w. Express your answer as an integer or as a common or improper fraction reduced to lowest terms.

Name: ______________________ Team Code: _______

2022 John O'Bryan Mathematical Competition
Junior/Senior Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. ______________________ **11.** ______________________

2. ______________________ **12.** ______________________

3. ______________________ **13.** ______________________

4. ______________________ **14.** ______________________

5. ______________________ **15.** ______________________

6. ______________________ **16.** ______________________

7. ______________________ **17.** ______________________

8. ______________________ **18.** ______________________

9. ______________________ **19.** ______________________

10. ______________________ **20.** ______________________

Name: ______ANSWERS______ **Team Code:** _______

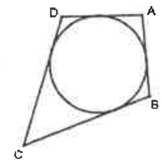
2022 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1. $\sqrt{15}$ — Must be this radical expression.
2. 14
3. 24
4. 3
5. $2\sqrt{61}$ — Must be this radical expression.
6. 2
7. 8
8. -15
9. $i-1$ or $\sqrt{-1}-1$ — Must be one of these.
10. 14
11. 18
12. 11
13. -3/2 — Must be this reduced fraction.
14. 5
15. 3560
16. 0 or zero
17. 5/2 — Must be this improper fraction.
18. 0.375 or .375 — Must be this decimal.
19. 2:5 — Must be this exact ratio.
20. 40/9 — Must be this reduced fraction.

2022 John O'Bryan Mathematical Competition
Questions for the Two-Person Speed Event

Calculators may not be used on the first four questions



1. A system of equations has $x^2 - 4y^2 = 30$ and $x - 2y = 5$ and $k = x + 2y$. Quadrilateral $ABCD$ is circumscribed about a circle with side lengths $BC = 20$ and $AD = 17$. The perimeter of the quadrilateral is w. Determine the value of $k + w$.

2. For the system of equations $\begin{cases} 3x - Ay = 3 \\ -4x + 5y = -1 \end{cases}$, let k be the value of A such that the system is inconsistent. Let w be the value of A such that this system represents perpendicular lines. Determine the product (kw). Express your answer as an integer or as a common or improper fraction reduced to lowest terms.

3. In degree mode, let $s(x) = \sin^{-1}(x)$, the inverse sine function) and let $t(x) = \tan(x)$. Determine the exact value of $s\left(t\left(s\left(t\left(s\left(t(30)\right)\right)\right)\right)\right)$. Do not include the "degree" symbol in your answer.

4. Let $k = \langle 2,3 \rangle \cdot \langle 4, -1 \rangle$ (the dot product of two vectors) and let $w = log_7(7203) - log_7(3)$. Determine the sum $(k + w)$.

Calculators may be used on the remaining questions

5. The sum of the lengths of two legs of a right triangle is 49. The square of the numeric length of the hypotenuse is 1225. Determine the numeric area of this triangle.

6. The volume of sphere with a radius of 9 is $k\pi$. The slope of a line passing through the point $(\frac{1}{2}, 56)$ is two times b, the y-intercept of this line. Determine the sum of $(k + b)$.

7. Hailey and Jay each toss a fair coin 5 times. Determine the probability that Hailey tossed at least 3 more heads then Jay. Express your answer as a common fraction reduced to lowest terms.

8. Let k and w be positive integers such that $1 + 2 + 3 + \cdots + k = k^2 - 15$ and $1 + 3 + 5 + 7 + \cdots + (2w - 1) = 2w^2 + 3w - 340$. Determine the sum $(k + w)$.

9. Let k be the number such that the reciprocal of half this number increased by half the reciprocal of this number is one-half. Monte Carlo travelled 120 miles in 105 minutes and then made the return trip along the same exact route at 60 miles per hour. Let w be Monte's average speed in miles per hour for the entire trip. Determine the sum $(k + w)$.

10. One dozen more than two dozen score, plus six times the square root of four. Divide that by seven, add three times eleven, get nine square and just a bit more. Determine how much more. Note: One score is 20.

Names: ______________________________ School: ______________________

2022 John O'Bryan Mathematical Competition
Two-Person Speed Event

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

SCORE

1. ______________________ ____

2. ______________________ ____

3. ______________________ ____

4. ______________________ ____

5. ______________________ ____

6. ______________________ ____

7. ______________________ ____

8. ______________________ ____

T1. ______________________

T2. ______________________

Calculators are not allowed to be used on the first four questions!

This competition consists of eight competitive rounds. Correct answers will receive the following scores:

1st: 7 points
2nd: 5 points
All Others: 3 points

There is a three minute time limit on each round. You may submit only one answer each round. To submit your answer, fold this sheet **lengthwise** and hold it high in the air so that a proctor may check your answer.

SCORE

Names: ______________________________ School: ______________________

2022 John O'Bryan Mathematical Competition
Answers for the Two-Person Speed Event

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

	Answer	SCORE
1.	80	____
2.	-9	____
3.	90 "Degrees" optional	____
4.	9	____
5.	294	____
6.	1000	____
7.	$\frac{7}{128}$ Must be this reduced common fraction	____
8.	23	____
T1.	69	
T2.	24	

Calculators are not allowed to be used on the first four questions!

This competition consists of eight competitive rounds. Correct answers will receive the following scores:

1st: 7 points
2nd: 5 points
All Others: 3 points

There is a three minute time limit on each round. You may submit only one answer each round. To submit your answer, fold this sheet **lengthwise** and hold it high in the air so that a proctor may check your answer.

SCORE